An Interior-point Algorithm for Mixed Complementarity Problems

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1. Abstract

Complementarity problems arise in mathematical models of several applications in Engineering, Economy and different branches of Physics. We mention contact problems and dynamic of multiple body systems in Solid Mechanics. In this paper we present a new feasible directions interior-point algorithm for mixed nonlinear complementarity problems that we have called FDA-MNCP. This algorithm is an extension of the FDA-NCP, an algorithm for complementarity problems recently proposed by the authors. The FDA-MNCP begins at any point strictly satisfying the inequality conditions and generates a sequence of interior points that converges to a solution of the problem. The sequence of iterates is generated in such a way that a suitable potential function is monotonically reduced. At each iterate, the algorithm finds a feasible direction, with respect to the region defined by the inequality conditions, which is also descent for the potential function. Then, an inexact line search along this direction is performed in order to define the next iterate. Results about global and asymptotic convergence for the FDA-MNCP algorithm are stated. Numerical results obtained with the proposed algorithm for several well known benchmark problems are presented. These results agree with the asymptotic analysis and show that the FDA-MNCP algorithm is efficient and robust.

2. Keywords: Feasible Direction Algorithm, Interior-point Algorithm, Mixed Nonlinear Complementarity Problems.

3. Introduction

We consider the Mixed Nonlinear Complementarity Problem, (MNCP):

Find \((x, y) \in \mathbb{R}^n \times \mathbb{R}^m\) such that

\[
x \geq 0, \quad F(x, y) \geq 0 \quad \text{and} \quad x \cdot F(x, y) = 0, \quad Q(x, y) = 0,
\]

where \(F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n\) and \(Q : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m\), are continuously differentiable and \(x \cdot F(x, y)\) represents a Hadamard product, means the product of coordinated

\[
\begin{pmatrix}
x_1 F_1(x, y) \\
\vdots \\
x_n F_n(x, y)
\end{pmatrix}.
\]

Let \(\Omega := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | x \geq 0, \ F(x, y) \geq 0\}\) the feasible set and \((x, y) \in \Omega\) a feasible point. A point at the interior of \(\Omega\), that we call \(\Omega^0\), is an interior point.

Several mathematical models in different disciplines, like engineering, physics and economics, lead to complementarity problems. This is the case of static and dynamic contact in solid mechanics, where there is a complementarity condition between the contact forces and the gap, see [Christensen et al. (1998)], [Petersson (1995)] and [Tanoh et al. (2004)]. Limit Analysis and Plasticity models can also include a complementarity condition, see [Zouain et al. (1993)] and [Tin-Loi (1999b)]. When this kind of mechanical models are employed for structural optimization, several authors employ mathematical programming problems with equilibrium constraints, MPEC, as in [Tin-Loi (1999a)]. Models for free boundary problems can also involve complementarity conditions, see [Leontiev et al. (2002)]. Optimality conditions of classical constrained optimization problems as well as bi-level programs, [Herskovits et al. (2000)], or Nash-Cournot equilibrium include a complementarity condition. Further applications of complementarity problems are described in [Ferris and Pang (1997)].

The MNCP can be also written as follows:

Find \((x, y) \in \mathbb{R}^n \times \mathbb{R}^m\) such that \(x \geq 0, \ F(x, y) \geq 0\) and
\[ S(x, y) = \begin{pmatrix} H(x, y) \\ Q(x, y) \end{pmatrix} = 0, \quad (2) \]

where \( H(x, y) = x \cdot F(x, y) \).

Considerable research effort was devoted by mathematicians and engineers to solve this problem, looking for strong and efficient techniques for real engineering applications.

Our approach is based on the iterative solution of of the nonlinear system (2). Given an initial feasible point, the present algorithm generates a sequence of feasible points such that the potential function

\[ f(x, y) = \phi(x, y) + \|Q(x, y)\|^2, \]

where \( \phi(x, y) = x^T F(x, y) \), is reduced at each iteration. At each iterate a search direction is computed. This one is a descent direction with respect to the potential function and a feasible direction of the problem. A line search procedure ensures that the new point is in fact feasible and the potential lower.

This algorithm follows the ideas of algorithm FDA_NCP described in [Herskovits and Mazorche (2008)].

4. Basic ideas of the present approach

We discuss now the basic ideas involved in the present technic. Let us consider the nonlinear system of equations (2) and the region \( \Omega_c = \{ (x, y) \in \Omega \mid f(x, y) \leq c \}, c \in \mathbb{R}_+ \), where seek the solutions.

Then, a Newton - Raphson iteration to solve (2) is given by the following expression:

\[ \nabla S(x^k, y^k) \left( \begin{array}{c} x^{k+1} - x^k \\ y^{k+1} - y^k \end{array} \right) = -S(x^k, y^k), \]

Instead of taking \( (x^{k+1}, y^{k+1}) \) computed above, we define a search direction as \( d^k = \left( \begin{array}{c} x^{k+1} - x^k \\ y^{k+1} - y^k \end{array} \right) \).

But this direction may not be feasible. Then, we introduce a perturbed Newton iteration and so we have

\[ \nabla S(x^k, y^k) d^k = \begin{pmatrix} -x^k \cdot F(x^k, y^k) + \rho_k E_1 \\ -Q(x^k, y^k) \end{pmatrix} \]

where, \( \rho_k = \frac{\alpha \phi(x^k, y^k)}{n} \in (0, 1) \), \( E_1 = \begin{bmatrix} 1, \ldots, 1^T \end{bmatrix} \in \mathbb{R}^n \), \( \alpha \in (0, 1) \), \( \beta \in [1, 2] \) and

\[ \nabla S(x^k, y^k) = \begin{pmatrix} \nabla_x H(x^k, y^k) & D_x \nabla_y F(x^k, y^k) \\ \nabla_x Q(x^k, y^k) & \nabla_y Q(x^k, y^k) \end{pmatrix} \]

for all \( (x^k, y^k) \in \Omega_c \) and \( D_x \in \mathbb{R}^{n \times n} \) is a diagonal matrix such that \( D_{ii,x} \equiv x_i \).

It is shown in [Mazorche (2007)] that \( d^k \) is a descent direction of the potential function \( f(x, y) \) and is a feasible direction in \( \Omega \). The vector \( d^k \) is a perturbation of Newton’s direction.

In effect, since \( d^k \) is a descent direction, we have that \( d^k \cdot \nabla f(x^k, y^k) < 0 \). We prove later that, taking \( \rho_k = \frac{\alpha \phi(x^k, y^k)}{n} \), it is

\[ \nabla f(x^k, y^k) \cdot d^k \leq -(1 - \rho_0 \phi(x^k, y^k)^{\beta-1}) f(x^k, y^k), \]

where \( 1 - \frac{\rho_k}{\phi(x^k, y^k)} > 0 \).

A new iterate is then obtained by performing an inexact line search procedure that looks for a new feasible point with a sufficient reduction of the potential function \( f(x, y) \). We employ an extension of Armijo’s line search that deals with inequality constraints, proposed by [Herskovits (1986)].

Line search procedures based on Wolfe’s or Goldstein’s inexact line search criteria can also be employed, [[Bazaraa and Shetty (1979)], [Herskovits (1995)], [Herskovits (1998)]]. These are more efficient in practise.

The present iterations were obtained by introducing a perturbation in Newton’s algorithm, that has quadratic rate of convergence. Then, working with a smaller perturbation will result in a faster convergence. Near a solution, we take \( \rho_k = O(\phi^3(x^k, y^k)) \) and discuss the cases, when \( \beta \in (1, 2) \) and \( \beta = 2 \).
4.1 FDA_MNCP: Feasible Directions Algorithm for MNCP
The present algorithm is stated as follows:

Parameters:
c > 0, α, η, ν ∈ (0, 1), β ∈ (1, 2] and ρ_0 < α \min\{1, 1/(c^β-1)\}.

Initial Data: \((x^0, y^0) \in Ω^0\) such that \(f(x^0, y^0) < c\) and \(k = 0\).

Step 1: Calculation of the search direction.
Compute \(d^k\) by solving:
\[
\nabla S(x^k, y^k)d^k = \begin{pmatrix}
-x^k \cdot F(x^k, y^k) + ρ^k E_1 \\
-Q(x^k, y^k)
\end{pmatrix},
\]
where \(ρ^k = ρ_0 \varphi(x^k, y^k)\).

Step 2: Line search.
Take \(d^k = \left(\begin{array}{c}
d^k_x \\
d^k_y
\end{array}\right)\). Compute \(t^k\), the first number of the sequence \(\{1, ν^2, ...\}\) satisfying
\[
x^k + t^k d^k_x ≥ 0 \quad (5)
\]
\[
F(x^k + t^k d^k_x, y^k + t^k d^k_y) ≥ 0 \quad (6)
\]
\[
f(x^k + t^k d^k_x, y^k + t^k d^k_y) ≤ f(x^k, y^k) + t^k η \nabla f(x^k, y^k)t d^k \quad (7)
\]

Step 3: Updates.
Set \((x^{k+1}, y^{k+1}) := (x^k + t^k d^k_x, y^k + t^k d^k_y)\) and \(k := k + 1\).

Go back to Step 1.

The present algorithm is very simple to implement and requires a computer effort similar to that of Newton method for nonlinear systems of equations.

5. Study of global convergence
In this section we define a set of assumptions about the MNCP and prove global convergence to a solution of the problem.

Assumption 1 The set \(Ω_c \equiv \{(x, y) \in Ω | f(x, y) ≤ c\}\) is a compact and has an interior \(Ω^0_c\). Each \((x, y) \in Ω^0_c\) satisfies \(x > 0\) and \(F(x, y) > 0\).

Assumption 2 The functions \(F(x, y)\) and \(Q(x, y)\) are of classes \(C^1(ℜ^n × ℜ^m)\), \(\nabla F(x, y)\) and \(\nabla Q(x, y)\) satisfies Lipschitz condition
\[
\|\nabla F(x_2, y_2) - \nabla F(x_1, y_1)\| ≤ γ_0 \| (x_2, y_2) - (x_1, y_1) \|
\]
and
\[
\|\nabla Q(x_2, y_2) - \nabla Q(x_1, y_1)\| ≤ L \| (x_2, y_2) - (x_1, y_1) \|,
\]
for any \((x_1, y_1), (x_2, y_2) \in Ω_c\), where \(γ_0\) and \(L\) are positive real numbers.

Assumption 3 The matrix \(\nabla S(x^k, y^k) = \begin{pmatrix}
\nabla_x H(x^k, y^k) & D_x \nabla_y F(x^k, y^k) \\
\nabla_x Q(x^k, y^k) & \nabla_y Q(x^k, y^k)
\end{pmatrix}\) has an inverse in \(Ω_c\).
**Assumption 4** There is constant real \( \sigma > 0 \) such that the following subset \( \Omega^* \) is not empty:

\[
\Omega^* \equiv \{(x, y) \in \Omega_c \text{ such that } \sigma\|Q(x, y)\| \leq \phi(x, y)\}.
\]

The assumption 3 and 4 implies that \( x \) and \( F(x, y) \) are not zero simultaneously for \((x, y) \in \Omega_c\). We also have that the linear system (4) has always a solution.

Since \( \nabla F(x^k, y^k) \) and \( \nabla Q(x^k, y^k) \) are continuous, we have that the matrix in the assumption 3 has an inverse continuous in \( \Omega_c \). Thus, there exists a scalar \( \kappa > 0 \) such that \( \|\nabla S(x^k, y^k)^{-1}\| \leq \kappa \) for any \((x^k, y^k) \in \Omega_c\).

The following results prove that the search direction of the present algorithm is bounded, is a descent direction and constitutes an uniformly feasible directions field. These are valid for \( \beta \in [1, 2] \).

**Lemma 1** For any \((x^k, y^k) \in \Omega_c\), the search direction \( d^k \) satisfies

\[
\|d^k\| \leq \overline{\kappa} \phi(x^k, y^k)
\]

In consequence, \( \|d^k\| \leq \overline{\kappa}_c \).

**Proof:**

Let be \((x^k, y^k) \in \Omega_c\) and \( \overline{E} = [E_1, 0, \ldots, 0] \). We have,

\[
\| - S(x^k, y^k) + \rho^k \overline{E} \| = \| x^k \cdot F(x^k, y^k) \|^2 - 2\rho^k \phi(x^k, y^k) + n(\rho^k)^2 + \|Q(x^k, y^k)\|^2.
\]

the same arguments in [Herskovits and Mazorche (2008)] and by assumption 4, we have

\[
\| - S(x^k, y^k) + \rho^k \overline{E} \| \leq \frac{1 + \sigma^2}{\sigma^2} \phi(x^k, y^k)^2.
\]

Considering now (4), we obtain bounds on \( \|d^k\|\):

\[
\|d^k\| \leq \pi - S(x^k, y^k) + \rho^k \overline{E},
\]

where \( \overline{\kappa} = \kappa \sqrt{\frac{1 + \sigma^2}{\sigma^2}} \). Thus, (8) follows from (9) and (10). In consequence, we have \( \|d^k\| \leq \overline{\kappa}_c \).  

**Lemma 2** The search direction \( d^k \) is a descent direction for \( f(x^k, y^k) \) at any \((x^k, y^k) \in \Omega_c\) such that \( x^k \cdot F(x^k, y^k) \neq 0 \).

**Proof:**

See in [Mazorche (2007)]

**Lemma 3** The search direction \( d^k \), given by the present algorithm, constitutes a uniformly feasible directions field of the problem for \((x^k, y^k) \in \Omega_c\).

**Proof:**

It follows from Assumption 2 that \( \nabla S(x, y) \) satisfies Lipschitz condition in \( \Omega_c \). Let be \( \gamma \) a positive real number such that, for all and \( 1 \leq i \leq n \),

\[
\|\nabla S_i(x^k_1, y^k_1) - \nabla S_i(x^k_2, y^k_2)\| \leq \gamma \|(x^k_2, y^k_2) - (x^k_1, y^k_1)\|
\]

for all \((x^k_1, y^k_1), (x^k_2, y^k_2) \in \Omega^* \).

Let be \((x^k, y^k) \in \Omega^* \) and \( \theta \) such that \([x^k, y^k], (x^k, y^k) + \tau d^k] \subset \Omega \) for \( \tau \in [0, \theta] \). If follows from the Mean Value Theorem that

\[
S_i((x^k, y^k) + \tau d^k) \geq S_i(x^k, y^k) + \tau \nabla S_i(x^k, y^k) d^k - \tau^2 \gamma \|d^k\|^2
\]

for any $\tau \in [0, \theta]$ and $i = 1, 2, \ldots, n$. Since
\[
\nabla S_i^T(x^k, y^k)d^k = -x_i^k F_i(x^k, y^k) + \rho^k 
\] it is
\[
S_i((x^k, y^k) + \tau d^k) \geq (1 - \tau)S_i(x^k, y^k) + (\rho^k - \tau \gamma\|d^k\|^2)\tau.
\]
Then, for $\tau \leq \min\{1, \frac{\rho^k}{\gamma\|d^k\|^2}\}$, it is
\[
S_i((x^k, y^k) + \tau d^k) \geq 0
\]
for $i = 1, 2, \ldots, n$. Considering now lemma 1, for $\rho^k$ defined in the algorithm, we have that this is also true for
\[
\tau \leq \min\{1, \frac{\rho_0 S_{\beta-2}}{\gamma n \kappa^2}\}
\] since $\beta \leq 2$, the present lemma is valid for
\[
\theta = \min\{1, \frac{\rho_0 c_{\beta-2}}{\gamma n \kappa^2}\}.
\]

**Lemma 4** There exists $\zeta > 0$ such that, for $x^k \in \Omega_c$, condition (7) is satisfied for any $t^k \in [0, \zeta]$.

Proof:

Let be $t^k \in (0, \theta]$, where $\theta$ was obtained in the previous lemma. Applying the Mean Value theorem for $i = 1, 2, \ldots, n$ and do $(x^{k+1}, y^{k+1}) = (x^k, y^k) + t^k d^k$, we have
\[
S_i((x^{k+1}, y^{k+1})) \leq S_i(x^k, y^k) + t^k \nabla[S(x^k, y^k)]d^k + t^k \gamma\|d^k\|^2.
\]
Summing the previous $n$ inequalities and considering equation (11), we get:
\[
\phi(x^{k+1}, y^{k+1}) \leq (1 - t^k)\phi(x^k, y^k) + t^k n \rho^k + n t^k \gamma\|d^k\|^2.
\] (13)
Similarly,
\[
Q_i^2(x^{k+1}, y^{k+1}) \leq (1 - t^k)Q_i^2(x^k, y^k) + t^k \gamma\|d^k\|^2.
\]
Summing the previous $m$ inequalities
\[
\|Q(x^{k+1}, y^{k+1})\|^2 \leq (1 - t^k)\|Q(x^k, y^k)\|^2 + m t^k \gamma\|d^k\|^2.
\] (14)
Summing 13 and 14,
\[
\phi(x^{k+1}, y^{k+1}) \leq (1 - t^k)\phi(x^k, y^k) + t^k (n \rho^k + (n + m) t^k \gamma\|d^k\|^2).
\]
Then
\[
\phi(x^{k+1}, y^{k+1}) \leq \|1 - (1 - \rho_0 \phi(x^k, y^k) - 1 - \frac{n + m}{1 + c \sigma^2} \phi(x^k, y^k) t^k) f(x^k, y^k)
\]
if (7) is satisfied. Thus, it is enough to have
\[
(1 - \rho_0 \phi(x^k, y^k) - 1 - \frac{n + m}{1 + c \sigma^2} \phi(x^k, y^k) t^k) \geq \eta(1 - \rho_0 \phi(x^k, y^k) - 1 - \frac{n + m}{1 + c \sigma^2} \phi(x^k, y^k) t^k)
\]
As in the previous lemma, we get
\[ t^k \leq (1 - \eta)(1 - \rho_0 \phi(x^k, y^k) \beta^{-1}) \frac{1 + c \sigma^2 \phi(x^k, y^k)}{n + m} \frac{1}{\gamma \|d^k\|^2} \] (15)

Then, the present lemma is true for
\[ \xi = \min \{(1 - \eta)(1 - \rho_0^\beta) \phi(x^k, y^k) \beta^{-1} + c \sigma^2 (n + m) \gamma \kappa^2 c, \theta\}, \]
where \( \theta \) was obtained in Lemma 3.

**Lemma 5** There exists \( \xi > 0 \) such that, for \( (x^k, y^k) \in \Omega^* \), the point \( (x^{k+1}, y^{k+1}) = (x^k, y^k) + t^k d^k \) belongs to set \( \Omega^* \) for any \( t^k \in [0, \xi] \).

**Proof:**
See in [Mazorche (2007)].

In consequence of the three previous lemmas, we deduce that the number of steps required by Armijo’s line search included in Step 2) of the algorithm is finite and bounded above. The following theorem proves global convergence of the present algorithm to a solution of the complementarity problem.

**Theorem 1** Given an initial feasible point, \( (x^0, y^0) \in \Omega^* \), the sequence \( \{(x^k, y^k)\} \) generated by the present algorithm converges to \((x^*, y^*)\), solution of problem (1).

**Proof.**
It follows from Lemmas 1 to 4 that \( (x^k, y^k) \in \Omega_c \). Since \( \Omega_c \) is a compact, \( \{(x^k, y^k)\} \) has accumulation points in \( \Omega_c \). Let be \((x^*, y^*)\) an accumulation point. Since the step length is always positive, we deduce that \( \|d^k\| \to 0 \). Considering (4), we have that \( \{f(x^k, y^k)\} \to 0 \). Thus, \((x^*, y^*)\) is a solution of the problem.

6. Study of asymptotic convergence

The search direction of the present algorithm is a perturbation of Newton’s iteration for nonlinear systems of equations. Including a line search, it can be proved that Newton’s method has quadratic convergence, provided that the step-length goes to one, [Dennis and Schnabel (1996)]. It is natural to expect better rates of convergence of our algorithm, for smaller values of \( \rho^k \).

As in Maratos effect, encountered in nonlinear constrained optimization, we cannot ensure that a unitary step-length is always obtained, see [Herskovits and Santos (1998)], [Herskovits et. al. (2005)]. In the present case, taking smaller values for \( \rho^k \) increases the possibility of having this effect.

**Theorem 2** Consider the sequence \( \{(x^k, y^k)\} \) generated by the present algorithm, that converges to a solution \((x^*, y^*)\) of (1). Then,

(i) Taking \( \beta \in (1, 2) \), \( t^k = 1 \) for \( k \) large enough, and the rate of convergence of the present algorithm is at least superlinear.
(ii) If \( t^k = 1 \) for \( k \) large enough and \( \beta = 2 \), then the rate of convergence is quadratic.

**Proof:**
Considering Theorem 1, we deduce from (12) and (15) that for \( k \) large enough, the step length obtained in Armijo’s line search is \( t^k = 1 \).
Standard Newton’s method analysis procedures can be used to show that
\[
\|(x^{k+1}, y^{k+1}) - (x^*, y^*)\| \leq (1 - t_k)\|(x^k, y^k) - (x^*, y^*)\| + \frac{\kappa \rho_0 \phi^\beta(x^k, y^k)}{\sqrt{n}} + O\|(x^k, y^k) - (x^*, y^*)\|, \tag{16}
\]
see [Dennis and Schnabel (1996)], for example.

From the mean value theorem and Lipschitz condition it follows
\[
\phi^\beta(x_1, y_1) \leq \phi^\beta(x_2, y_2) + \epsilon((x_1, y_1) - (x_2, y_2)) \text{ for some } \epsilon \in (0, 1) \text{. Taking } (x_2, y_2) = (x^*, y^*) \text{,}
\]

Taking \((x_2, y_2) = (x^*, y^*)\), for all \((x_1, y_1) = (x^k, y^k)\) sufficiently near \((x^*, y^*)\) it is
\[
\phi^\beta(x^k, y^k) \leq \phi^\beta((x, y))\beta \sqrt{n} O\|(x^k, y^k) - (x^*, y^*)\|.
\]

(i) Then, for \(\beta \in (1, 2), \phi^\beta(x^k, y^k) = o\|(x^k, y^k) - (x^*, y^*)\|\). By substitution in (16) we get
\[
\lim_{k \to \infty} \frac{\|(x^{k+1}, y^{k+1}) - (x^*, y^*)\|}{\|(x^k, y^k) - (x^*, y^*)\|} = 0.
\]

Thus, the rate of convergence is superlinear.

(ii) The result for \(\beta = 2\) is obtained in a similar way.

Introducing a line search along an arc, as in ([Herskovits and Santos (1998)], [Herskovits et. al. (2005)]), it seems possible to avoid Maratos effect, even in the case when \(\beta = 2\).

7. Numerical study

To study the numerical behavior of the present algorithm in practical applications we solve a set of test problems largely employed in mathematical programming literature.

We take \(\rho_0 = \alpha \min[1, \phi^{\beta-1}(x^k, y^k)]\) in the numerical implementation. In this way, extremely large deflections are avoided far of the solution and \(\rho_0\) is constant when \(\phi(x^k, y^k)\) is small. We study two cases, for \(\beta = 1.1\) and \(\beta = 2\). All the problems were solved with the same set of parameters, \(\alpha = 0.25, \eta = 0.4\) and \(\nu = 0.8\). The present algorithm stops when \(f(x^k, y^k) < 10^{-8}\), for all the examples. "Iter" represents the main iterations to solve the problem for the given stopping criteria and "Iter LS", the total number of extra function evaluations required by Armijo’s line search. The collection of tests is in [Mazorche (2007)]

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7.1 Modeling the contact problem with the BEM

The classical form of the Signorini’s problem in linear elasticity reads as follow:

\begin{align*}
\textbf{a) } -\nabla \cdot \sigma &= \bar{f} \quad \text{on } \Omega \\
\textbf{b) } u &= \bar{u} \quad \text{in } \Gamma^D \\
\textbf{c) } p &= \bar{p} \quad \text{in } \Gamma^N \\
\textbf{d) } u \cdot \bar{n} + \bar{s} &\geq 0 \quad \text{in } \Gamma^C \\
\textbf{e) } p \cdot \bar{n} &\geq 0 \quad \text{in } \Gamma^C \\
\textbf{f) } (u \cdot \bar{n} + \bar{s}) \cdot (p \cdot \bar{n}) &= 0
\end{align*}

where \( \Omega \) is the open domain occupied by the solid and \( \Gamma = \Gamma^D \cup \Gamma^N \cup \Gamma^C \) its boundary, \( u \) is the displacement function, the Cauchy stress tensor \( \sigma = \mathcal{C} \varepsilon \), \( \varepsilon = \nabla^S u \) with \( \mathcal{C} \) the elasticity tensor and \( \nabla^S u = 1/2(\nabla u + \nabla^T u) \). Function \( p = \varepsilon n \) with \( n \) the outward unit normal vector of \( \Gamma \). Functions \( \bar{f}, \bar{u}, \bar{p}, \bar{n} \) and \( \bar{s} \) are given (see Fig. 1).

Figura 1: Contact problem in linear elasticity

The boundary integral equation for linear elasticity with \( \bar{f} = 0 \) in Eq. 17 is [Brebbia et al (1984), Beer and Watson (1992), París and Cañas (1998)]:

\[ c(\xi)u(\xi) = \int_{\Gamma} u^*(\xi, x)p(x) \, d\Gamma - \int_{\Gamma} p^*(\xi, x)u(x) \, d\Gamma \]  

(18)

where function \( u^* \) is the fundamental solution for the linear elasticity problem and \( p^* \) is its correspondent fundamental surface traction. Matrix \( c(\xi) \) depends on the local geometry of boundary \( \Gamma \) at point \( \xi \) and the second integral on the right is defined in the Cauchy principal value sense [Brebbia et al (1984), Beer and Watson (1992), París and Cañas (1998)].

Applying the BEM method, the discrete form of the boundary integral equation result:

\[ Hu - Gp = 0 \]  

(19)

where now, the vectors \( u \) and \( p \) define the displacements and traction forces on the boundary \( \Gamma \) and \( H \) and \( G \) are the BEM matrices.

Applying the boundary conditions Eqs. (17.b) and (17.c) to Eq. 19, and denoting \( x \) the vector of unknowns related to the normal tractions in \( \Gamma^C \) and \( y \) the vector of remaining unknowns, we obtain:

\[ Ax - By = q \]  

(20)

The boundary conditions in \( \Gamma^C \), Eqs. (17.d) to (17.f), can be written as:

\[ S(y) \geq 0 \]
\[ x \geq 0 \]
\[ S(y) \cdot x = 0 \]  

(21)
Finally, defining $Q(x, y) = Ax + By - q$, we have the following Mixed Complementarity Problem:

\[
\begin{align*}
S(y) & \geq 0 \\
x & \geq 0 \\
S(y) \cdot x & = 0 \\
Q(x, y) & = 0
\end{align*}
\]  \hspace{1cm} (22)

This example consists of a curved beam in contact with a rigid plane as shown by Fig. 2. This three-dimensional problem was solved using the BEM.

Figure 2: Curved beam.

Figure 3 shows the obtained contact region. The points with positive contact force are enhanced in red color showing the characteristic elliptic shape of the contact region.

Figure 3: Pressures in the contact region of the curved beam.

8. Conclusions

In this paper a new feasible points algorithm for mixed nonlinear complementarity problem was presented. Global convergence was proved and two theoretical results about the asymptotic convergence were obtained. Taking $\beta \in (1, 2)$, superlinear convergence was strongly proved. When $\beta = 2$, a weaker proof of quadratic convergence was obtained. The performance of the present algorithm in the numerical examples was very good. The present technique is simple and also robust, since there are not significant parameters to be adjusted and all the test problems were solved with the same set of parameters.
We also remark that conditioning of the linear system (4) is improved by the fact that all the iterates are strictly feasible.

Even if Maratos effect was rarely observed in the test problems, including a line search along an arc should improve the efficiency and robustness of the present method in a similar way as in [Herskovits et. al. (2005)], [Herskovits and Santos (1998)].

Our results can also be extended to other problems, like Mathematical Program with Equilibrium Constraints(MPEC).

9. References


[Tanoh et al. (2004)] Tanoh G, Renard Y and Noll D (2004), Computational Experience With An Interior Point Algorithm For Large Scale Contact Problems, optimization Online.

