MATERIAL DISTRIBUTION AND SIZING OPTIMIZATION OF FUNCTIONALLY GRADED PLATE-SHELL STRUCTURES

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Abstract: A high order shear deformation theory is used to develop a discrete model for the structural and sensitivity analyses allowing for material distribution and sizing optimization of functionally graded material structures. The finite element formulation for general FGM plate-shell type structures is presented, and a non-conforming triangular flat plate/shell element with 24 degrees of freedom for the generalized displacements is used. The formulations accounts for geometric and material nonlinear behaviour, free vibrations and linear buckling analyses, and their analytical gradient based sensitivities. The p-index of the power law distribution and the thickness are the design variables. Mass, displacement, fundamental frequency and critical load are the objective functions or constraints. The solution of some illustrative plate-shell examples are presented and discussed.

Keywords: Functionally Graded Materials, Finite Element, Optimization.

1. INTRODUCTION

In an effort to develop the super heat resistant materials, Kozumi [1] first proposed the concept of Functionally Graded Material (FGM). Typical FGM plate-shell type structures are made of materials which are characterized by a continuous variation of the material properties over the thickness direction by mixing two different materials, metal and ceramic. The metal–ceramic FGM plates and shells are widely used in aircraft, space vehicles, reactor vessels, and other engineering applications. Structures made of composite materials have been widely used to satisfy high performance demands. In such structures stress singularities may occur at the interface between two different materials. In contrast, in FGM plate-shell structures the smooth and continuous variation of the properties from one surface to the other eliminates abrupt changes in the stress and displacement distributions. In certain applications, structures can experience large elastic deformations and finite rotations. Geometric nonlinearity plays a significant role in the behaviour of a plate or shell structures, especially when it undergoes large deformations. Also material nonlinearity has a significant role in the behaviour of these structures. Research in FGM structures has been done in the recent years. The authors, Moita et al. [2] have published recently a paper where static nonlinear analysis is investigated. Also a work in linear buckling has been presented, Moita et al. [3]. In these works, the authors call for a significant number of references that had been used for the formulation and comparing results. For this reason, here we cite few of them: Reddy and Arciniega [4] presented free vibration analysis of FGM plates. The same authors [5] carried out the large deformation analysis of FGM shells. Kim et al. [6] presented the geometric nonlinear analysis of functionally graded material plates and shells using a four-node quasi-conforming shell element. Zhao and Liew [7] analyzed the nonlinear response of functionally graded ceramic–metal shell panels under mechanical and thermal loading, using a displacement field expressed in terms of a set of mesh-free kernel particle functions. Ramu and Mohanty [8] applied the finite element method, using the classical plate theory, for the modeling and buckling analysis of FGM rectangular plates, under uniaxial and biaxial compression loads along with simply supported boundary conditions. The mechanical properties of FGM are dependent of the p-index of the power-law that defines the volume fraction of ceramic phase. Thus, the FGM structures could be designed to meet the specific requirements of each particular application. Accurate and efficient structural analysis, design sensitivity analysis and optimization procedures are fundamental to accomplish this task. The objective of this work is to present the development of analytical sensitivities for FGM structures, following the work of Moita et al. [9], and based on the previous structural analysis works, [2] and [3], and to present the optimization of some plate-shell structures using a gradient based algorithm.

2. FORMULATION OF FGM MODEL

An FGM is made by mixing two distinct isotropic material phases, for example a ceramic and a metal. The material properties of an FGM plate/shell structures are assumed to change continuously throughout the thickness, according to the volume fraction of the constituent materials. Power-law function [10] and exponential function [11] are commonly used to describe the variations of material properties of FGM. However, in both power-law and exponential functions, the stress concentrations appear in one of the interfaces in which the material is continuously but rapidly changing. Therefore, Chung and Chi [12] proposed a sigmoid FGM, which was composed of two power-law functions to define a new volume fraction. Chi and Chung [13] indicated that the use of a sigmoid FGM can significantly reduce the stress intensity factors of a cracked body. To describe the volume fractions, the power-law function is used here.
2.1 Power-law function: P-FGM
The volume fraction of the ceramic phase is defined according to the power-law [10]. In the present work, the continuous variation of the materials mixture is approximated by the using a certain number of virtual layers throughout the thickness direction - layer approach. In this sense, equations can be written for each virtual layer as follows:

\[ V_c^k = \left( 0.5 + \frac{z}{h} \right)^p ; \quad V_m^k = 1.0 - V_c^k \]  

(1)

where \( z \) is the thickness coordinate of mid-surface of each layer.

Once the volume fraction \( V_c^k \) and \( V_m^k \) have been defined, the material properties (H) of each virtual layer of an FGM can be determined by the rule of mixtures.

\[ H^k = V_c^k H_c + V_m^k H_m \]  

(2)

where \( H \) stands for Young’s modulus \( E \), the Poisson’s ratio \( \nu \), the mass density \( \rho \), or any other mechanical property.

Figure 1 show the variation of Young’s modulus \( E \) through the thickness, obtained using the sigmoid function and a 20 layer approach.

3. Displacement field, strains, and constitutive relations for FGM structures
The present theory considers large displacements with small strains. The used displacement field is based on the Reddy’s third-order shear deformation theory [14]:

\[
\begin{align*}
    u(x, y, z) &= u_0(x, y) - z \frac{\partial y_0}{\partial x} + z^3 c_1 \left[ -\frac{\partial w_0}{\partial x} \right] \\
    v(x, y, z) &= v_0(x, y) + z \frac{\partial x_0}{\partial y} + z^3 c_1 \left[ -\frac{\partial w_0}{\partial y} \right] \\
    w(x, y, z) &= w_0(x, y)
\end{align*}
\]  

(3)

The stress-strain relations for each layer \( k \), and corresponding linear elastic constitutive equation, can be written as follows:

\[ \sigma_k = Q_k \varepsilon_k ; \quad \dot{\sigma}_k = \dot{D}_k \varepsilon_k^0 \]  

(4)
where $\sigma_k = [\sigma_x, \sigma_y, \sigma_{xy}, \tau_{xz}, \tau_{yz}]^T$ is the stress vector and $\varepsilon_k = [\varepsilon_x, \varepsilon_y, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}]^T$ is the strain vector. $Q_k$ is the elasticity matrix, and $\varepsilon_k$ are the resultant forces and moments, and $\bar{\varepsilon}_k$ the constitutive matrix.

In this work, geometrically non-linear behaviour of FGM plate/shell type structures under mechanical loading is also considered. To the geometrically nonlinear behaviour, the Green’s strain tensor is here considered. Its components are conveniently represented in terms of the linear and nonlinear parts of the strain tensor. The linear strain components associated with the displacement field given above, can be represented in a synthetic form:

$$\varepsilon_k^l = \left\{ \begin{array}{c} 0 \\
 \frac{e_{mk}}{e_{kk}} + \frac{Z_k}{e_{kk}} + Z_k^3 \frac{e_{3kk}}{e_{kk}} \\
 \frac{\varepsilon_{xk}}{e_{kk}} + \frac{Z_k}{e_{kk}} \frac{\varepsilon_{33k}}{e_{kk}} \end{array} \right\} \quad (5)$$

4. Finite Element Formulation

In the present works it is used a non-conforming triangular plate/shell finite element model having three nodes and eight degrees of freedom per node: the displacements $u_0$, $v_0$, $w_0$, the slopes $\theta_{0x}$, $\theta_{0y}$, $\theta_{0z}$, the rotation $\theta_{0i}$, the strains $\varepsilon_{xk}$, $\varepsilon_{yk}$, $\varepsilon_{zk}$, the slopes $b_k$, and the rotations $\theta_{ki}$. The rotation $\theta_{ki}$ is introduced to consider a fictitious stiffness coefficient $K_{0z}$ to eliminate the problem of a singular stiffness matrix for general shape structures [15]. The element local displacements, rotations and slopes, are expressed in terms of nodal variables through shape functions $\{N_i\}$ given in terms of area co-ordinates $L_{ij}$, [15]. The displacement field can be represented in matrix form as:

$$u = Z(\sum_{i=1}^3 N_i d_i) = ZNa \quad ; \quad d = \sum_{i=1}^3 N_i d_i = Na$$

where $a$ and $d$ are the element and nodal displacement vectors, respectively, and $Z$ an appropriate matrix.

The membrane, bending and shear strains, as well the higher order bending and shear strains can be represented by:

$$\varepsilon_m^b = B^m a \quad ; \quad \varepsilon_b = z B^b a \quad ; \quad \varepsilon_s^b = B^s a \quad ; \quad \varepsilon_{eb} = z^3 B^{eb} a \quad ; \quad \varepsilon_{es} = z^2 B^{es} a$$

where $B^m$, $B^b$, $B^{eb}$, $B^s$, $B^{es}$, are components of the strain-displacement matrix $B$, and are given explicitly in [16].

4.1 Equations for nonlinear, buckling and free vibration analyses.

The development of these equations can be observed in references [2] and [9], and thus we have for nonlinear analysis:

$$\left( \begin{array}{cc} K_L & + K_\sigma \\ + K_\sigma & \end{array} \right) \left( \begin{array}{c} \varepsilon_{L} \\ \varepsilon_{\sigma} \end{array} \right) = \left( \begin{array}{c} F_{ext} \\ - \left( \varepsilon_{int} \right) \end{array} \right) i^{-1}$$

For linear buckling analysis, we make use of only one load increment, and equation (8) takes the following form:

$$(K_L + K_\sigma)\theta = F_{ext} \quad ; \quad \theta = \lambda \theta$$

For free vibrations we have:

$$(K_L - \omega^2 M) \nu = 0$$

In the previous equations, the definition of the matrices and vectors are given by:

$$K_L = \int_{A^e} \sum_{k=1}^N \bar{\varepsilon}_k T \{D_k\} B dA^e \quad ; \quad K_\sigma = \int_{A^e} \sum_{k=1}^N \bar{\varepsilon}_k T \{G_k\} G T dA^e \quad ; \quad F_{ext} = \int_{A^e} \sum_{k=1}^N \bar{\varepsilon}_k T \{i_k\} i \nu T dA^e \quad ; \quad F_{int} = \int_{A^e} \sum_{k=1}^N \bar{\varepsilon}_k T \{D_k\} \nu T dA^e$$

$$M = \int_{A}^N \sum_{k=1}^N \{N_i\} T \{D_k\} \{N_i\} dA^e$$
5. Sensitivity analysis in P-FGM structures

From the power-law, equation (1), the material properties (H) of each layer of an FGM can be determined by the rule of mixtures:

\[ H^k = V_c^k (H_c - H_m) + H_m \]  \hspace{1cm} (13)

Differentiating in order of the design variables \( b \), comes:

\[ \frac{dH^k}{db} = \frac{dV_c^k}{db} (H_c - H_m) \quad ; \quad \frac{dV_c(z)}{db} = \frac{d}{db} \left( 0.5 + \frac{1}{h} z_k \right)^p \]  \hspace{1cm} (14)

For design variable \( p \)-index, we have

\[ \frac{dH^k}{dp} = \frac{dV_c(z)}{dp} (H_c - H_m) = \left( 0.5 + \frac{1}{h} z_k \right)^p \ln \left( 0.5 + \frac{1}{h} z_k \right) (H_c - H_m) \]  \hspace{1cm} (15)

and for design variable thickness \( h \):

\[ \frac{dH^k}{dh} = \frac{dV_c(z)}{dh} (H_c - H_m) = -p \left( 0.5 + \frac{1}{h} z_k \right)^{p-1} \left( \frac{z_k}{h^2} \right) (H_c - H_m) \]  \hspace{1cm} (16)

5.1 Linear analysis

Differentiating \( K_L \) \( \mathbf{q} = \mathbf{F}_{\text{ext}} \) comes

\[ \frac{dq}{db} = -K_L^{-1} \left( \frac{\partial K_L}{\partial b} \right) \mathbf{q} \]  \hspace{1cm} (17)

The element stiffness matrix can be written in the form

\[ K_L = \int B^T A \sum_{k=1}^{N} Q_k \frac{z_{k+1} - z_k}{n} B \ dA \quad \text{where} \quad \mathbf{D} = \int \sum_{k=1}^{N} Q_k \frac{z_{k+1} - z_k}{n} \ dA \]  \hspace{1cm} (18)

and its derivatives, comes as follows:

\[ \frac{\partial K_L}{\partial b} = \int B^T A \sum_{k=1}^{N} \frac{\partial Q_k}{\partial b} \frac{z_{k+1} - z_k}{n} B \ dA \]  \hspace{1cm} (19)

\[ \frac{\partial K_L}{\partial h} = \int B^T A \sum_{k=1}^{N} \frac{\partial Q_k}{\partial h} \frac{z_{k+1} - z_k}{n} B \ dA + \int B^T A \sum_{k=1}^{N} Q_k \frac{\partial}{\partial h} \left( \frac{z_{k+1} - z_k}{n} \right) B \ dA \quad \text{n=0,1,2,3,4,5,7} \]  \hspace{1cm} (20)

5.2 Nonlinear analysis

The equilibrium equation for nonlinear analysis at the final of each increment, can be written as [9]:

\[ \mathbf{F}_{\text{int}} (\mathbf{q}, \mathbf{b}) = \mu \mathbf{F}_{\text{ext}} \]  \hspace{1cm} (21)

Differentiating this equation in order to the design variables comes:

\[ \frac{dq}{db} = -K_L^{-1} \left( \mu \frac{\partial \mathbf{F}_{\text{ext}}}{\partial b} - \frac{\partial \mathbf{F}_{\text{int}}}{\partial b} \right) \]  \hspace{1cm} (22)

with

\[ \frac{\partial \mathbf{F}_{\text{int}}}{\partial b} = \int B^T A \left[ \frac{\partial D}{\partial b} \right] \epsilon^T \ dA = \int B^T A \sum_{k=1}^{N} \frac{\partial Q_k}{\partial b} \frac{z_{k+1} - z_k}{n} \epsilon^T \ dA + \int B^T A \sum_{k=1}^{N} Q_k \frac{\partial}{\partial b} \left( \frac{z_{k+1} - z_k}{n} \right) \epsilon^T \ dA \]  \hspace{1cm} (23)
5.3 Fundamental frequency
From the equation for free vibrations (10), we obtain, after development, the sensitivity of the fundamental frequency

\[
\frac{d\omega}{db} = \frac{1}{2\omega} v^T \left( \frac{\partial K_L}{\partial b} - \omega^2 \frac{\partial M}{\partial b} \right) v
\]

(24)

5.3.1 Fundamental frequency constraint
Defining the constraint by \( \Psi = 1 - \alpha/\omega_0 \), its sensitivity comes as follows:

\[
\frac{d\Psi}{db} = -\frac{1}{2\omega_0 \alpha_0} v^T \left( \frac{\partial K_L}{\partial b} - \omega^2 \frac{\partial M}{\partial b} \right) v
\]

(25)

where the derivatives of element mass matrix, equation (12), are given by:

\[
\frac{\partial M}{\partial p} = \int_{A}^{N} \sum_{k=1}^{N} h_k^{1/2} z^T z \, dz \, \frac{\partial \rho_k}{\partial p} 
\]

(26)

\[
\frac{\partial M}{\partial h} = \int_{A}^{N} \sum_{k=1}^{N} h_k \rho_k \frac{\partial (Z^T Z)}{\partial h} 
\]

(27)

5.4 Critical load
From the equation (9), we obtain, after development, the sensitivity of critical load parameter \( \lambda_{cr} \)

\[
\frac{d\lambda_{cr}}{db} = v^T \left( \frac{\partial K_L}{\partial b} - \lambda_{cr} \frac{\partial K_{\alpha}}{\partial b} \right) v
\]

(28)

5.4.1 Critical load constraint
As for fundamental frequency, considering the constraint \( \Psi = \lambda_{cr} / \lambda_{cr0} - 1 \leq 0 \), we obtain

\[
\frac{d\Psi}{db} = \frac{1}{\lambda_{cr0} \lambda_{cr}} v^T \left( \frac{\partial K_L}{\partial b} - \lambda_{cr} \frac{\partial K_{\alpha}}{\partial b} \right) v = 0 \quad \text{with} \quad \frac{\partial K_{\alpha}}{\partial b} = \int_{A}^{N} \frac{\partial G}{\partial b} G \, dA
\]

(29)

5.6 Effective stress constraint
From the Huber-Mises criterion, for the case of an isotropic material the effective stress \( \bar{\sigma} \) is given by:

\[
\bar{\sigma}_k = \left[ \left( \sigma_{11}^k \right)^2 + \left( \sigma_{22}^k \right)^2 - \left( \sigma_{11}^k \sigma_{22}^k \right) + 3 \left( \sigma_{12}^k \right)^2 + 3 \left( \sigma_{13}^k \right)^2 + 3 \left( \sigma_{23}^k \right)^2 \right]^{1/2}
\]

(30)

and the constraint is defined as follows

\[
\Psi = \frac{\bar{\sigma}_k}{(\sigma_Y)_k} = \left[ \left( \sigma_{11}^k \right)^2 + \left( \sigma_{22}^k \right)^2 - \left( \sigma_{11}^k \sigma_{22}^k \right) + 3 \left( \sigma_{12}^k \right)^2 + 3 \left( \sigma_{13}^k \right)^2 + 3 \left( \sigma_{23}^k \right)^2 \right]^{1/2}
\]

(31)

\[
^t \sigma_{m,b}^k = Q_k \cdot \epsilon_k = Q_k \left( B^m + z B^b + z^2 B^w \right) a
\]

\[
^t \sigma_{s}^k = Q_k \cdot \epsilon_k = Q_k \left( B^s + z B^w \right) a
\]

(32)

and \( t \) means current time (at the end of the final increment).

The sensitivity of effective stress constraint is then given by:
are obtained by the sensitivity analysis performed using the

\[
\frac{\partial^2 \sigma_1}{\partial \rho^2} \left( \sigma_{m0} + \sigma_k \right) \left( \sigma_{q0} \right) \left( \sigma_{a0} + \sigma_k \right) = Q_k \frac{\partial^2 \sigma_1}{\partial \sigma^2} \left( \sigma_{m0} + \sigma_k \right) \left( \sigma_{q0} \right) \left( \sigma_{a0} + \sigma_k \right)
\]

where the necessary derivatives are given in the Appendix.

6. Optimization

In the present work the objective of optimization is to find the design variables that minimize or maximize an objective function, subject to or not to constraints:

\[
\min \phi(q, b) \text{ or } \max \phi(q, b)
\]

subjected to \( \hat{\nu}^2(q, b) \leq \nu, \ i=0,1,..n \)

and \( b_l \leq b \leq b_u \)

where \( b_l \) and \( b_u \) are the lower and upper limits of the design variables.

This optimization problem is solved using the gradient based algorithm FAIPA, the Feasible Arc Interior Point Algorithm, Herskovits et al. [18].

7. Applications

7.1 Simply supported square FGM plate.

A simply-supported square FGM plate with constituents zirconia (\( E_z = 151 \text{ GPa} \), \( \rho_z = 3000 \text{ kg/m}^3 \), \( v_z = 0.3) \), and stainless steel (\( E_s=200.7877 \text{ GPa} \), \( \rho_s = 8166 \text{ kg/m}^3 \), \( v_s = 0.31776 \)), side-to-thickness ratio \( a/h=60 \), is considered. The analytical sensitivity accuracy is performed for this application, with \( p\)-index=5.0. Considering first a static nonlinear analysis, the sensitivities of central displacement \( w_c \) and of effective stress \( \bar{\sigma} \) are obtained at the end of the incremental-iterative process, \( F=125 \text{ kN/m}^2 \). Considering next a free-vibration analysis we obtain the sensitivity of fundamental frequency. Results obtained by analytical sensitivities and by global finite differences have a very good agreement. The results obtained for mass, centre displacement and fundamental frequency as a function of the power law exponent \( p \) (varying between 0.1 and 10.0) show that the minimum mass and the maximum fundamental frequency are obtained for \( p=0.1 \), but the minimum centre displacement is obtained for \( p=10.0 \).

First we consider an initial plate with exponent \( p=1.0 \) to which corresponds a mass \( m=20.1 \text{ kg} \). The centre displacement and effective stress in nonlinear behaviour, and the fundamental frequency, are obtained. For load level \( F=125 \text{ kN/m}^2 \), corresponding to 3 increments, the centre displacement and effective stress are \( w_c=3.72 \text{ mm} \) and \( \bar{\sigma} = 95.7 \text{ MPa} \), respectively. The fundamental frequency of the plate is \( \omega_0=948.25 \text{ rad/s} \). The optimization consists in searching the power law exponent \( p \) that minimizes the mass of the plate considering the following constraints: maximum center displacement...
The final result is achieved for $p=0.22$ with corresponding mass $m=14.1$ kg.

Table 1. Optimal design results for the simply supported FGM plate.

<table>
<thead>
<tr>
<th>Initial plate</th>
<th>Variable $p$</th>
<th>$m$ (kg)</th>
<th>$w_c$ (mm)</th>
<th>$\sigma$ (MPa)</th>
<th>$\omega$ (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>20.1</td>
<td>3.7</td>
<td>141.7</td>
<td>948.2</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Final plate</th>
<th>Constraint $w_c$</th>
<th>Constraint $\sigma$</th>
<th>Constraint $\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.22</td>
<td>$\leq 4.0$</td>
<td>$\leq \sigma_Y$</td>
<td>$\geq 10000$</td>
</tr>
<tr>
<td>14.1</td>
<td>4.0 *</td>
<td>140.2</td>
<td>1080.0</td>
</tr>
</tbody>
</table>

* active constraint

7.2 Hinged functionally graded cylindrical panel.

A P-FGM cylindrical shell panel represented in Figure 2 has the straight sides simply-supported (hinged) and the curved sides free. The geometry is defined by: $R=2.54$ m, $L=0.508$ m and subtended angle $2\Theta=0.2$ rad, $h=0.0126$ m, and it is modelled by a $8 \times 8$ finite element mesh. The constituents are zirconia and aluminium with $E_c=151$ GPa, $\rho_c=3000$ kg/m$^3$, $E_m=70$ GPa, $\rho_m=2707$ kg/m$^3$, $v_c=v_m=0.3$, and a power law exponent $p=5$ obtaining a cylindrical panel with mass $m=8.96$ kg. The panel is subjected to a center point reference load $F_0=60$ kN. In geometrically nonlinear analysis using an incremental-iterative process, a center displacement $w_c=7.96$ mm is obtained with 6 increments ($F=51.0$ kN). The optimization consists, in the 1st level of optimization, to obtain minimum center displacement as a function of the power law exponent $p$, the design variable, which for this application has side constraints $0.2 < p < 10$.

In the 2nd level of the optimization the design variable is now the thickness of the panel, and the minimum value for the mass of the panel is searched considering de maximum center displacement constraint $W_{max} \leq 4.0$ mm . Table 2 describes completely the optimization process.

![Cylindrical panel](image)

Figure 2. Cylindrical panel

Table 2. Optimal design results for the hinged-free FGM cylindrical panel.

<table>
<thead>
<tr>
<th>Cylindrical Panel</th>
<th>Initial</th>
<th>Optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^{st}$ level of optimization</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variable $p$</td>
<td>5.0</td>
<td>0.2</td>
</tr>
<tr>
<td>Mass $m$ (kg)</td>
<td>8.96</td>
<td>9.60</td>
</tr>
<tr>
<td>$\phi$ (objective) – $w_c$ (mm)</td>
<td>7.96</td>
<td>3.40</td>
</tr>
<tr>
<td>$2^{nd}$ level of optimization</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variable $h$ (mm)</td>
<td>12.60</td>
<td>11.78</td>
</tr>
<tr>
<td>Constraint $w$ (mm)</td>
<td>3.4</td>
<td>4.0 *</td>
</tr>
<tr>
<td>$\phi$ (objective) – $m$ (kg)</td>
<td>9.60</td>
<td>8.97</td>
</tr>
</tbody>
</table>

* active constraint

Considering the panel divided into 4 groups of finite elements for the second level of the optimization, as shown in Figure 3, the final panel is found for the values of the project variables given in Table 3, corresponding to the mass $m=7.80$ Kg.

However, in the previous study, we have not take into account the possibility of the plate reached the yield stress, and thus to consider the yield stress as a constraint. To allow for this constraint we need to do a geometric (G) and materially (M) nonlinear analysis. The panel is subjected, as previously, to a centre point reference load $F_0=60$ kN, and a G+M nonlinear analysis is done, using an incremental-iterative process. A centre displacement $w_c=11.6$ mm is obtained with 6 increments ($F=51$ kN). In the 1st level of optimization, as previously the power law exponent obtained is $p=0.2$. 

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In the 2nd level of the optimization, the design variable is now the thickness of the plate, and the minimum value for the mass of the panel is searched considering the following constraints: maximum centre displacement $W_{\text{max}} \leq 4.0 \text{mm}$, and maximum effective stress (at metal surface) $\bar{\sigma} \leq \sigma_{Y_{\text{FGM}}}$. Table 4 describes completely the optimization process.

Table 3. Optimal design results for the hinged-free FGM cylindrical panel using 4 groups of finite elements

<table>
<thead>
<tr>
<th>2nd level of optimization</th>
<th>φ (objective) - m (kg)</th>
<th>7.80</th>
<th>9.60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constraint w (mm)</td>
<td>3.4</td>
<td>4.0 *</td>
<td></td>
</tr>
<tr>
<td>Variables $h_i$ (mm)</td>
<td>12.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Element Group</td>
<td></td>
<td>7.68</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>7.91</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>10.11</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>15.27</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 3. Initial and final panels

Table 4. Optimal design results for the hinged-free FGM cylindrical panel.

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<thead>
<tr>
<th>Cylindrical Panel</th>
<th>Initial</th>
<th>Optimal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st level of optimization</td>
<td></td>
</tr>
<tr>
<td>Variable $h$ (mm)</td>
<td>12.60</td>
<td>12.0</td>
</tr>
<tr>
<td>Mass m (kg)</td>
<td>8.96</td>
<td>9.60</td>
</tr>
<tr>
<td>φ (objective) - $w_c$ (mm)</td>
<td>11.60</td>
<td>3.40</td>
</tr>
<tr>
<td>2nd level of optimization</td>
<td></td>
<td>2nd level of optimization</td>
</tr>
<tr>
<td>Variable $h$ (mm)</td>
<td>12.60</td>
<td>12.0</td>
</tr>
<tr>
<td>Constraint $w = 4.0$ (mm)</td>
<td>3.40</td>
<td>4.0 *</td>
</tr>
<tr>
<td>Constraint $\bar{\sigma} = 285.0$ (MPa)</td>
<td>284.0</td>
<td></td>
</tr>
<tr>
<td>φ (objective) - m (kg)</td>
<td>9.60</td>
<td>9.15</td>
</tr>
</tbody>
</table>

Considering the panel divided into 4 groups of finite elements for the second level of the optimization, the final panel is found for the values of the project variables given in Table 5, corresponding to the mass $m=8.68$ Kg. Figure 4 shows the load-displacement paths for the initial plate, intermediate and final plate.

Table 5. Optimal design results for the hinged-free FGM cylindrical panel using 4 groups of finite elements

<table>
<thead>
<tr>
<th>2nd level of optimization</th>
<th>φ (objective) - m (kg)</th>
<th>8.68</th>
<th>9.60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constraint $w = 4.0$ (mm)</td>
<td>3.40</td>
<td>4.00 *</td>
<td></td>
</tr>
<tr>
<td>Constraint $\bar{\sigma} = 285.0$ (MPa)</td>
<td>272.13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variables $h_i$ (mm)</td>
<td>12.60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Finite Element Group</td>
<td>1</td>
<td>9.40</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>11.16</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>12.34</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>12.68</td>
<td></td>
</tr>
</tbody>
</table>
8. CONCLUSIONS.
The results obtained show that the analytical sensitivities with respect to p-index and thickness prove to be efficient, by comparison with the sensitivities calculated by global finite differences, which had been used to solve the same problems. Also the optimization process performed by FAIPA, Feasible Arc Interior Point Algorithm, show to be a good tool to obtain the optimal design of functionally graded plate and shell structures.

9. ACKNOWLEDGEMENTS: This work was supported by FCT, Fundaçao para a Ciencia e Tecnologia, through IDMEC, under LAETA, project UID/EMS/50022/2013, and also CNPq, CAPES and FAPERJ, from Brazil.

10. REFERENCES
APPENDIX

From equation (14), we define the matrix \( m = Z^T Z \), which non-zero terms are given by:

\[
m_{11} = m_{22} = m_{33} = z_{k+1} - z_k ; \quad m_{15} = -m_{24} = -c_2 \left( \frac{z_{k}^4 - z_{k+1}^4}{4} \right) ; \quad m_{17} = -m_{26} = -\frac{z_{k}^2 - z_{k+1}^2}{2} + c_2 \frac{z_{k}^4 - z_{k}^2}{4} ;
\]
\[
m_{44} = m_{55} = c_2 \left( \frac{z_{k}^7 - z_{k+1}^7}{7} \right) ; \quad m_{46} = m_{57} = c_2 \left( \frac{z_{k}^5 - z_{k+1}^5}{5} - c_2 \frac{z_{k}^7 - z_{k+1}^7}{7} \right) ; \quad m_{66} = m_{77} = \frac{z_{k}^3 - z_{k+1}^3}{3} - 2c_2 \frac{z_{k}^5 - z_{k}^3}{5} - c_2 \frac{z_{k}^7 - z_{k}^5}{7}
\]

For FG material structures, the vectorial distances for each \( k \) virtual layer, can be given by the following equations:

\[
z_{k+1}^n = \tilde{r}_{k+1}^n \frac{h}{2} ; \quad z_k = r_k \frac{h}{2} \quad \text{or} \quad \tilde{r}_{k+1}^n = \frac{2z_{k+1}^n}{h} ; \quad \tilde{r}_k = \frac{2z_k}{h}
\]

Thus we can write:

\[
\frac{z_{k+1}^n - z_k^n}{n} = \left( \frac{r_{k+1}^n - r_k^n}{n} \right) \left( \frac{h}{2} \right)^n
\]

and the corresponding derivatives

\[
\frac{\partial z_{k+1}^n - \partial z_k^n}{\partial h} = \left[ \frac{r_{k+1}^n - r_k^n}{2} \right] \frac{h}{2} = \left[ \frac{r_{k+1}^n - r_k^n}{2} \right] \frac{1}{2} ; \quad \frac{\partial z_{k+1}^n - \partial z_k^n}{\partial \tilde{r}_k} = \left[ \frac{2}{k+1} - \frac{2}{k} \right] \frac{1}{2} \frac{h}{2} ; \quad \frac{\partial z_{k+1}^n - \partial z_k^n}{\partial \tilde{r}_{k+1}^n} = \left[ \frac{2}{k+1} - \frac{2}{k} \right] \frac{1}{2} \frac{h}{2}
\]

\[
\frac{\partial z_{k+1}^n - \partial z_k^n}{\partial h} = \left[ \frac{3}{k+1} - \frac{3}{k} \right] \frac{h}{2} = \left[ \frac{3}{k+1} - \frac{3}{k} \right] \frac{1}{2} \frac{h}{2} ; \quad \frac{\partial z_{k+1}^n - \partial z_k^n}{\partial \tilde{r}_k} = \left[ \frac{4}{k+1} - \frac{4}{k} \right] \frac{1}{4} \frac{h}{2} ; \quad \frac{\partial z_{k+1}^n - \partial z_k^n}{\partial \tilde{r}_{k+1}^n} = \left[ \frac{4}{k+1} - \frac{4}{k} \right] \frac{1}{4} \frac{h}{2}
\]

\[
\frac{\partial z_{k+1}^n - \partial z_k^n}{\partial h} = \left[ \frac{5}{k+1} - \frac{5}{k} \right] \frac{h}{2} = \left[ \frac{5}{k+1} - \frac{5}{k} \right] \frac{1}{2} \frac{h}{2} ; \quad \frac{\partial z_{k+1}^n - \partial z_k^n}{\partial \tilde{r}_k} = \left[ \frac{6}{k+1} - \frac{6}{k} \right] \frac{1}{6} \frac{h}{2} ; \quad \frac{\partial z_{k+1}^n - \partial z_k^n}{\partial \tilde{r}_{k+1}^n} = \left[ \frac{6}{k+1} - \frac{6}{k} \right] \frac{1}{6} \frac{h}{2}
\]

\[
\frac{\partial z_{k+1}^n - \partial z_k^n}{\partial h} = \left[ \frac{7}{k+1} - \frac{7}{k} \right] \frac{h}{2} = \left[ \frac{7}{k+1} - \frac{7}{k} \right] \frac{1}{2} \frac{h}{2} ; \quad \frac{\partial z_{k+1}^n - \partial z_k^n}{\partial \tilde{r}_k} = \left[ \frac{7}{k+1} - \frac{7}{k} \right] \frac{1}{7} \frac{h}{2} ; \quad \frac{\partial z_{k+1}^n - \partial z_k^n}{\partial \tilde{r}_{k+1}^n} = \left[ \frac{7}{k+1} - \frac{7}{k} \right] \frac{1}{7} \frac{h}{2}
\]

or in a general form:

\[
\frac{\partial}{\partial h} \left( \frac{z_{k+1}^n - z_k^n}{n} \right) = \frac{r_{k+1}^n - r_k^n}{n} \frac{h}{2}
\]

Also the distance to the midle plan of the \( k \) virtual layer can be written in the form:

\[
\tilde{z}_k = \frac{z_{k+1} + z_k}{2} = \frac{\tilde{r}_k}{2} \frac{h}{2} \quad \text{or} \quad \tilde{z}_k = \frac{2\tilde{r}_k}{h} = \frac{z_{k+1} + z_k}{h}
\]

Thus for the derivatives we have:

\[
\frac{\partial \tilde{z}_k}{\partial h} = \frac{\tilde{r}_k}{2} ; \quad \frac{\partial (\tilde{z}_k)^2}{\partial h} = \frac{\tilde{r}_k^2}{h} ; \quad \frac{\partial (\tilde{z}_k)^3}{\partial h} = \frac{3\tilde{r}_k^3 h^2}{8}
\]